

STRATIFICATION AND STORM SURGES ALONG A STRAIGHT COAST,
WITH APPLICATIONS TO THE WESTERN COAST OF NORWAY

by

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Abstract

The theory of storm surges along a straight coast in case of a continuously stratified sea is considered.

The stratification is shown to have no appreciable effect on the storm surge, which therefore is mainly a barotropic response. The internal response which depends strongly on stratification, is found to be extremely sensitive to how the tangential stress depends on depth. Moreover, there is argued that the baroclinic response is substantially influenced by bottom friction, whereas the barotropic response is not. A solution based on this assumption is presented.

1. Introduction

Storm surges along a straight coast are generated when longshore winds with the coast to the right (Northern Hemisphere) forces water towards the coast (Ekman, 1905). For a review see Welander (1961), Heaps (1965) and Bretschneider (1967). In many respects the theory of storm surges parallels that of upwelling, which is the corresponding internal response (Smith, 1968).

However, while the internal response is strongly influenced by stratification (c.f. Charney, 1955), the present paper shows that the stratification has only a minor effect on the storm surge. Thus the storm surge in stratified seas may to a good approximation be derived from the concomitant problem with a homogeneous body of water. (See e.g. Gjevik and Røed, 1976, hereafter referred to as GR). Furthermore, if the storm surge in case of stratification, is derived from the concomitant homogeneous problem, it will correspond to a certain distribution of the tangential stress, which, in turn, determines the internal response. This might be important for upwelling problems since the internal response is rather sensitive to changes in the vertical distribution of the stress.

The ocean model applied is one with a constant depth and with continuous stratification. The motion is given by normal modes representation (Gill and Clarke, 1974). For negligible bottom friction, this representation is purely formal as far as the internal response is concerned, whereas for the surface response the series may be truncated after a few modes. For the baroclinic modes, therefore, a modified set of equations which includes bottom friction, are suggested and solved.

2. Formulation

Consider a continuously, stratified incompressible fluid. The motion will be described relatively to a Cartesian coordinate system with z -pointing upwards and with origin in the surface of the initially undisturbed fluid. The fluid is bounded by the bottom, $z = H(x,y)$, and the surface, $z = \eta_s(x,y,t)$, where η_s is initially zero. Moreover, the fluid is bounded by a straight coast along the x -axis, y pointing seawards. The stratification is horizontally uniform such that in the equilibrium state the density, $Q(z)$, and the pressure, $P(z)$, are functions of depth only. Thus the hydrostatic balance reads

$$\frac{dP}{dz} = - gQ(z) .$$

The surface is exposed to pressure and wind stress fields which are distributed over a horizontal domain with linear dimensions large compared to the depth. This entails that the hydrostatic approximation is justified. The disturbances are assumed to be sufficiently small for the motion to be described by linear theory. Thus the governing equations may be written

$$(2.1) \quad \frac{\partial \vec{v}}{\partial t} + f \vec{k} \times \vec{v} = - \frac{1}{Q_s} \nabla p + \frac{1}{Q_s} \frac{\partial \vec{\tau}}{\partial z}$$

$$(2.2) \quad \nabla \cdot \vec{v} + \frac{\partial}{\partial z} \left(\frac{\partial \eta}{\partial t} \right) = 0$$

$$(2.3) \quad \frac{\partial p}{\partial z} = - \sigma g$$

$$(2.4) \quad \frac{\partial \sigma}{\partial t} + \frac{\partial \eta}{\partial t} \frac{dQ}{dz} = 0$$

where use has been made of the kinematic condition

$$(2.5) \quad w = \frac{\partial \eta}{\partial t}.$$

$f = 2 \Omega \sin$ (latitude) is the Coriolis parameter, g the gravitational acceleration, $\vec{v} = u\vec{i} + v\vec{j}$ and w the horizontal and vertical part of the velocity, respectively, p the perturbation pressure and σ the perturbation density. Q_s is the surface value of the equilibrium density, and the Boussinesq approximation has been made use of. The vector, $\vec{\tau}$, is the tangential stress acting between horizontal planes whose value at the surface, $\vec{\tau}_s$, is the prescribed wind stress.

Eqs. (2.1) - (2.4) will be solved by the method of division into normal modes (Lighthill, 1969), and the presentation follows closely that of Gill and Clarke (loc.cit.) who, employed the technique in a coastal upwelling problem. The aim of the method is to separate off the horizontal and vertical variations in order to obtain equations in two dimensions only. To this end the variables are expanded as follows

$$(2.6) \quad \vec{v}(x, y, z, t) = \sum_{n=0}^{\infty} \vec{v}_n(x, y, t) \hat{\eta}'_n(z)$$

$$(2.7) \quad \eta(x, y, z, t) = \sum_{n=0}^{\infty} \eta_n(x, y, t) \hat{\eta}_n(z)$$

$$(2.8) \quad p(x, y, z, t) = p_s(x, y, t) + Q_s \sum_{n=0}^{\infty} p_n(x, y, t) \hat{\eta}'_n(z)$$

where p_s is the surface pressure. The eigenfunctions, $\hat{\eta}_n(z)$, may be shown to be solutions of the eigenvalue problem

$$(2.9) \quad \hat{\eta}''_n + \frac{N^2(z)}{c_n^2} \hat{\eta}_n = 0 \quad ; \quad -H < z < 0$$

$$(2.10) \quad \hat{\eta}'_n - \frac{g}{c_n^2} \hat{\eta}_n = 0 \quad ; \quad z = 0$$

$$(2.11) \quad \hat{\eta}_n = 0 \quad ; \quad z = -H .$$

p_n and η_n are found to be proportional, the factor of proportionality being the eigenvalue, c_n , squared,

$$(2.12) \quad p_n = c_n^2 \eta_n .$$

$N^2(z)$ is the Väisälä-Brunt frequency defined by

$$(2.13) \quad N^2 = - \frac{g}{Q_s} \frac{dQ}{dz} .$$

The eigenfunctions, which are orthogonal, may conveniently be normalized as follows

$$(2.14) \quad \int_{-H}^0 [\hat{\eta}'_n(z)]^2 dz = H .$$

The equations for the modes may now be shown to take the form (c.f. Gill and Clarke)

$$(2.15) \quad \frac{\partial \vec{v}_n}{\partial t} + f \vec{k} \times \vec{v}_n = - c_n^2 \nabla \eta_n + \vec{\tau}_n ,$$

$$(2.16) \quad \nabla \cdot \vec{v}_n + \frac{\partial \eta_n}{\partial t} = 0$$

Here, $\vec{\tau}_n$, is given by

$$(2.17) \quad \vec{\tau}_n = - \frac{\hat{\eta}_n(0)}{Q_s H} \nabla p_s + \frac{1}{Q_s H} \int_{-H}^0 \frac{\partial \vec{\tau}}{\partial z} \hat{\eta}'_n(z) dz ,$$

and represents the forcing.

In order to solve Eqs. (2.15) and (2.16) a knowledge of the variation of $\vec{\tau}$ with depth is required, or, if the stress is put proportional to the gradient of the horizontal velocity, \vec{v} , through an eddy viscosity coefficient, the variation with depth of this coefficient is required.

Furthermore, for application purposes the eigenfunctions and eigenvalues and hence the stratification must be known. A discussion of the eigenfunctions and eigenvalues associated with the chosen stratification may be found in the appendix. Different approaches relevant to the problem at hand may besides Gill and Clarke be found in Csanady (1972) and Mork (1972). In accordance with these investigations it is found (c.f. the appendix) that the zeroth or barotropic mode corresponds approximately to

$$(2.18) \quad c_0 = \sqrt{gH} \quad , \quad \hat{n}_0 = z + H .$$

The equations describing the response of the concomitant homogeneous problem may be found in GR. A comparison between Eq. (1) of GR and the barotropic mode equations (Eqs. (2.15) and (2.16) with $n = 0$) demonstrates that the barotropic mode necessarily must approximate the response to the concomitant problem (c.f. Mork).

The solution to Eqs. (2.15) and (2.16) are subject to the condition at the coast

$$v_n = 0 \quad ; \quad y = 0$$

and the initial conditions

$$\eta_n = 0 \quad \text{and} \quad \vec{v}_n = 0 \quad ; \quad t = 0 .$$

3. Stratification and tangential stress models

The area of application will be the western coast of Norway, where the hydrographic data have been regularly collected since the late twenties. Based on these observations the following stratification model is chosen: A well mixed layer of thickness 10 m with uniform density overlies a pycnocline layer of thickness 40 m in which the density increases linearly with depth from the surface value, $\rho_s = 1.0252 \text{ g/cm}^3$, to the value $\rho_B = 1.0275 \text{ g/cm}^3$, which is constant throughout the deep bottom layer. (Fig. 3.1.

The variation of the tangential stress, $\tau(z)$, with depth is more difficult to ascertain, mostly due to the turbulent character of the motion.

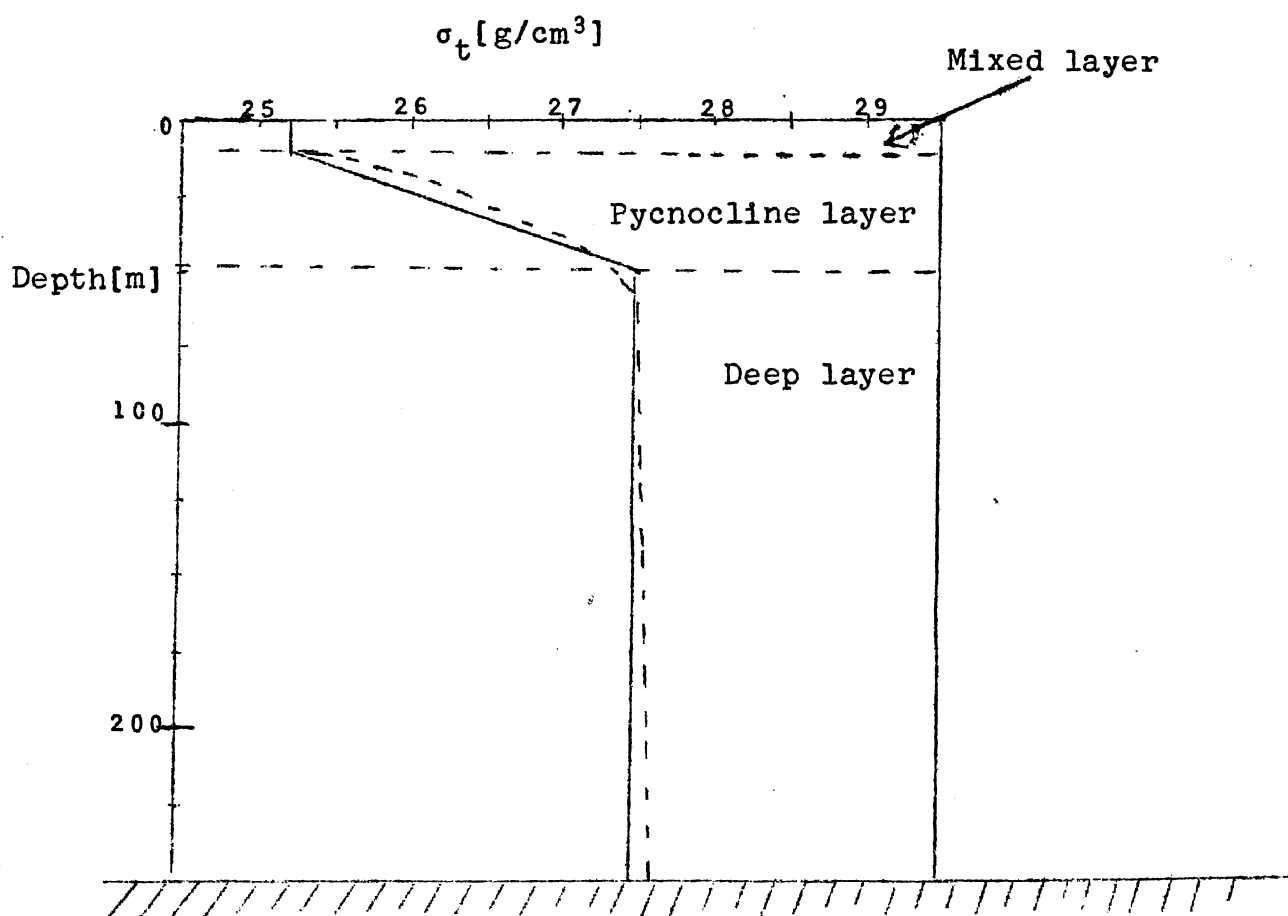


Fig.3.1 "Typical" stratification along the western coast of Norway. Dashed line is observed stratification.

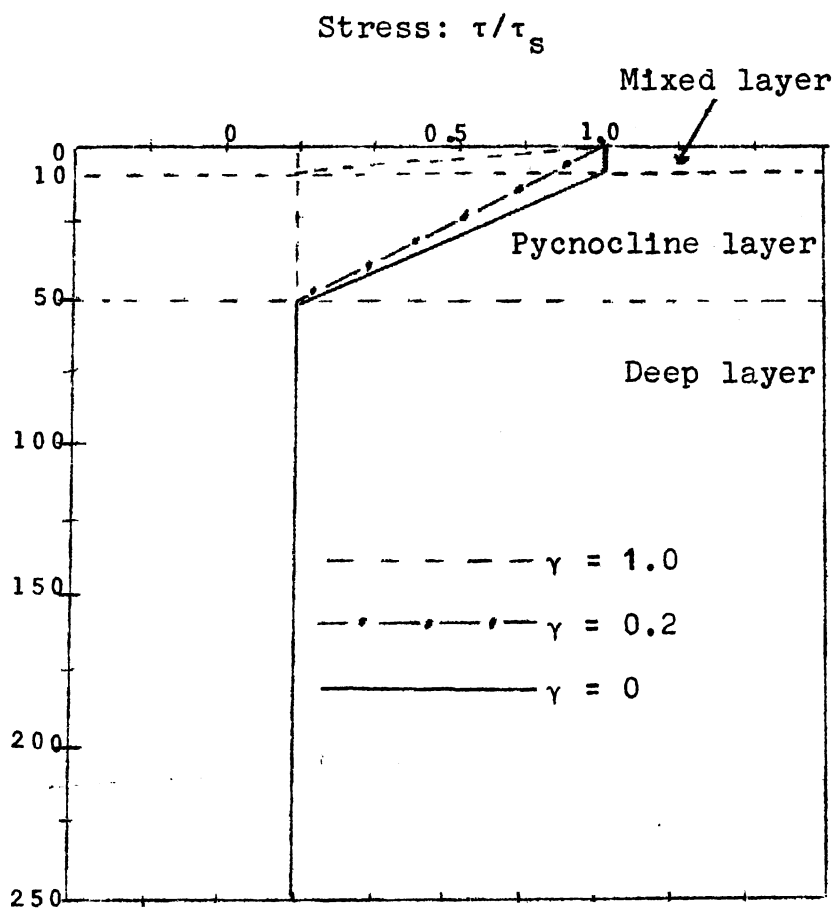


Fig. 3.2 The stress models applied

Three models will be considered.

- 1) A stress which decreases linearly from its surface value to zero at the bottom of the well mixed layer.
- 2) A stress which falls off linearly from its surface value to zero at the bottom of the pycnocline layer.
- 3) A stress which is constant and equal the surface value throughout the well mixed layer and subsequently decays linearly to zero at the bottom of the pycnocline layer. (Fig. 3.2).

The first model is one suggested by Gill and Clarke and applied by Csanady in his model of the Great Lakes. The second roughly corresponds to that applied in two layer models such as those of Charney and Veronis and Stommel (1956), whereas the third may be derived from the commonplace representation of the stress, viz.

$$(3.1) \quad \vec{\tau}(z) = \mu \frac{\partial \vec{v}}{\partial z},$$

where the eddy viscosity coefficient, μ , might be put inversely proportional to the density gradient, viz.

$$(3.2) \quad \mu = \lambda \frac{Q^2}{dQ/dz}.$$

Such a stress model has been used by Fjeldstad (1964) in a theoretical study of the frictional damping of free internal waves and later by Mork. If the mean velocity in the well mixed layer is uniform as suggested by Pollard and Millard (1970), the stress according to Eqs. (3.1) and (3.2) has to be constant throughout this layer. Within the pycnocline the representation (3.1) predicts the stress to be some function of depth and to be constant again in the deep layer, which due to the small level of turbulence may be put equal to zero, except, perhaps near the bottom. (Bottom friction).

Among the papers referred to in this section only Mork has included bottom friction in the form

$$(3.3) \quad \vec{\tau}_B = \mu \left. \frac{\partial \vec{v}}{\partial z} \right|_{z=-H} = \lambda Q_B \vec{v} \Big|_{z=-H} \quad (\lambda = \text{const.})$$

Invoking the representations (3.1) and (3.2), this leads to modes equations similar to Eqs. (2.15) and (2.16) save for that $\partial/\partial t$ in Eq. (2.15) is substituted by $\partial/\partial t + \lambda \frac{gH}{c_n^2}$. Since c_n is

proportional to n^{-1} for large n , he argues that at least the higher modes will be suppressed by friction.

4. Application to the passage of a single storm

In order to compare the stratified problem with the concomitant homogeneous one given by GR, the forcing is chosen as a moving storm, with no surface pressure fluctuations, viz.

$$(4.1) \quad \vec{\tau}_s = Q_s \tau_s e^{-\kappa(x-u_0 t)^2} \vec{1} ; \quad p_s = \text{const.}$$

The tangential stress, $\vec{\tau}(z)$, may be represented by

$$(4.2) \quad \vec{\tau} = Q_s \tau(z) e^{-\kappa(x-u_0 t)^2} \vec{1}$$

where

$$(4.3) \quad \tau(0) = \tau_s .$$

u_0 is the propagation speed of the storm, and $\kappa^{-\frac{1}{2}}$ measures the horizontal extent of the storm, i.e. small values of κ corresponds to wind fields of large extent. Invoking Eq. (2.17), the forcing function, $\vec{\tau}_n$, may be written

$$(4.4) \quad \vec{\tau}_n = \tau_n e^{-\kappa(x-u_0 t)^2} \vec{1}$$

where

$$(4.5) \quad \tau_n = \frac{1}{H} \int_{-H}^0 \frac{\partial \tau}{\partial z} \hat{n}'_n(z) dz .$$

With this definitions Eqs. (2.15) and (2.16) correspond to Eq. (1) of GR subject to similar boundary and initial conditions and they

may therefore be solved by the same procedure. Thus if the horizontal extent of the wind field, represented by $\kappa^{-\frac{1}{2}}$, is greater than or equal to the barotropic radius of deformation, $\frac{c_0}{f}$, the solution to Eqs. (2.15) and (2.16) in terms of the displacement of each mode, η_n , at the coast may to a good approximation be written *) [c.f. GR Eq. (17)]

$$(4.6) \quad \eta_n = \frac{1}{2} \frac{\tau_n}{c_n^2} \sqrt{\frac{\pi}{\kappa}} \frac{c_n}{c_n - u_0} \left[\operatorname{erf} \sqrt{\kappa} (x - u_0 t) - \operatorname{erf} \sqrt{\kappa} (x - c_n t) \right] ; \quad y=0$$

The three stress models of section 3 may conveniently be condensed in the expression

$$(4.7) \quad \tau(z) = \tau_s \begin{cases} 1 + \gamma \frac{z}{h_1} & ; \quad -h_1 \leq z \leq 0 \\ (1-\gamma) \frac{z+h_2}{h_2-h_1} & ; \quad -h_2 < z < -h_1 \\ 0 & ; \quad -H \leq z \leq -h_2 . \end{cases}$$

Model (1), (2) and (3) then corresponds to $\gamma = 1$, $\gamma = h_1/h_2$ and $\gamma = 0$, respectively. Substituting the expression for $\tau(z)$, Eq.(4.9), into Eq. (4.5) there results by invoking expression (A.2) for the eigenfunctions that

$$(4.8) \quad \tau_n = \tau_s \frac{c_n^2}{gH} \frac{s_n(\gamma)}{\hat{\eta}_n(0)}$$

where

$$(4.9) \quad s_n(\gamma) = \frac{\hat{g}\hat{\eta}_n(0)}{c_n^2} \left\{ \gamma \hat{\eta}_n'(0) + \frac{1-\gamma}{h_2-h_1} \left[\hat{\eta}_n(-h_1) - \hat{\eta}_n(-h_2) \right] \right\} .$$

*) If $\sqrt{\kappa} c_0 / f \leq 1$ then according to appendix $\sqrt{\kappa} c_n / f \ll 1$ for $n \geq 1$.

Thus from Eq. (2.7) the following expression for the displacement at the coast is obtained :

$$(4.10) \quad \eta = \frac{1}{2} \frac{\tau_s}{gH} \sqrt{\frac{\pi}{\kappa}} \sum_{n=0}^{\infty} \frac{c_n s_n(\gamma) \hat{\eta}_n(z)}{(c_n - u_0) \hat{\eta}_n(0)} \left[\operatorname{erf} \sqrt{\kappa} (x - u_0 t) - \operatorname{erf} \sqrt{\kappa} (x - c_n t) \right]$$

In table 1 are listed values of c_n and $s_n(\gamma)$ for the three stress models pertaining to the first six modes

Mode	n	c_n (m/s)	$s_n(\gamma)$		
			$\gamma = 1$	$\gamma = 0.2$	$\gamma = 0$
Barotropic	0	50.0435	0.9965	0.9945	0.9975
1st baroclinic	1	0.6850	8.4223	30.5128	4.2784
2nd baroclinic	2	0.2380	5.5210	-13.6181	-3.3333
3rd baroclinic	3	0.1385	2.9825	0.8774	0.1177
4th baroclinic	4	0.0964	1.7112	-0.8754	-0.8895
5th baroclinic	5	0.0735	1.0743	0.2257	0.0126

Table 1 : (H = 250 m, $h_1 = 10$ m, $h_2 = 50$ m) Eigenvalues, c_n , and amplitudes $s_n(\gamma)$. Note the peak for $s_n(\gamma)$ in the first baroclinic.

The solution (4.10) is purely formal unless the series can be truncated after a moderate number of terms. From the appendix there follows that for $n \geq 5$, $c_n s_n(\gamma)/(c_n - u_0)$ dies off as n^{-3} , whereas $\hat{\eta}_n(z)/\hat{\eta}_n(0)$ increases as n^2 unless z is close to zero. This fact together with the values listed in table 1 ensures that a very good approximation to the storm surge, $\eta_s = \eta(z=0)$ is obtained if the series is truncated after 6-7 terms, whereas for the internal displacements the solution (4.10) is a formal solution. Note, however, that for $y \neq 0$ the solutions might be applicable, c.f. Gill and Clarke (Loc.cit.).

Fig. 4.1 demonstrates both the concomitant homogeneous surge [Eq. (4.12) with $c_n = \sqrt{gH} \delta_{no}$, $s_n(\gamma) = \delta_{no}$ and $z=0$] and the surges in case of stratification for the three different stress models, respectively, calculated from the first 7 modes. As anticipated the zeroth or barotropic mode are in all three cases indistinguishable from the concomitant homogeneous surge. The baroclinic modes for which the first is the most dominant (c.f. table 1) provide a slight amplification of the maximum surge (within 5-10%). However, there is no time lag between the maxima. Moreover, for $ft > 5$ a nonvanishing surge is left in the stratified cases, which contrasts the concomitant homogeneous surge which rapidly tends to zero.

A main shortcoming of the present model so far is the neglect of bottom friction. In spite of this the effect of stratification on the storm surge is small. Obviously the inclusion of bottom friction will tend to reduce the surge. Especially as will be argued, this is true for the baroclinic modes. Thus the conclusion is that stratification effects on storm surges do not seem to be appreciable and for prediction purposes, therefore, homogeneous models will do. This conclusion is supported by the remarkable good agreement between observed surges and surges estimated from homogeneous models both analytically (GR; Heaps, 1965) and numerically (Heaps, 1969 and 1971).

Suppose, in case of stratification, that the storm surge should be given from the concomitant homogeneous problem. Thus the quantity that now may be determined is the stress amplitude $\tau(z)$, or rather, since its surface value is fixed, its variation with depth. However, no attempt will be made in the present investigation to find this distribution, but the

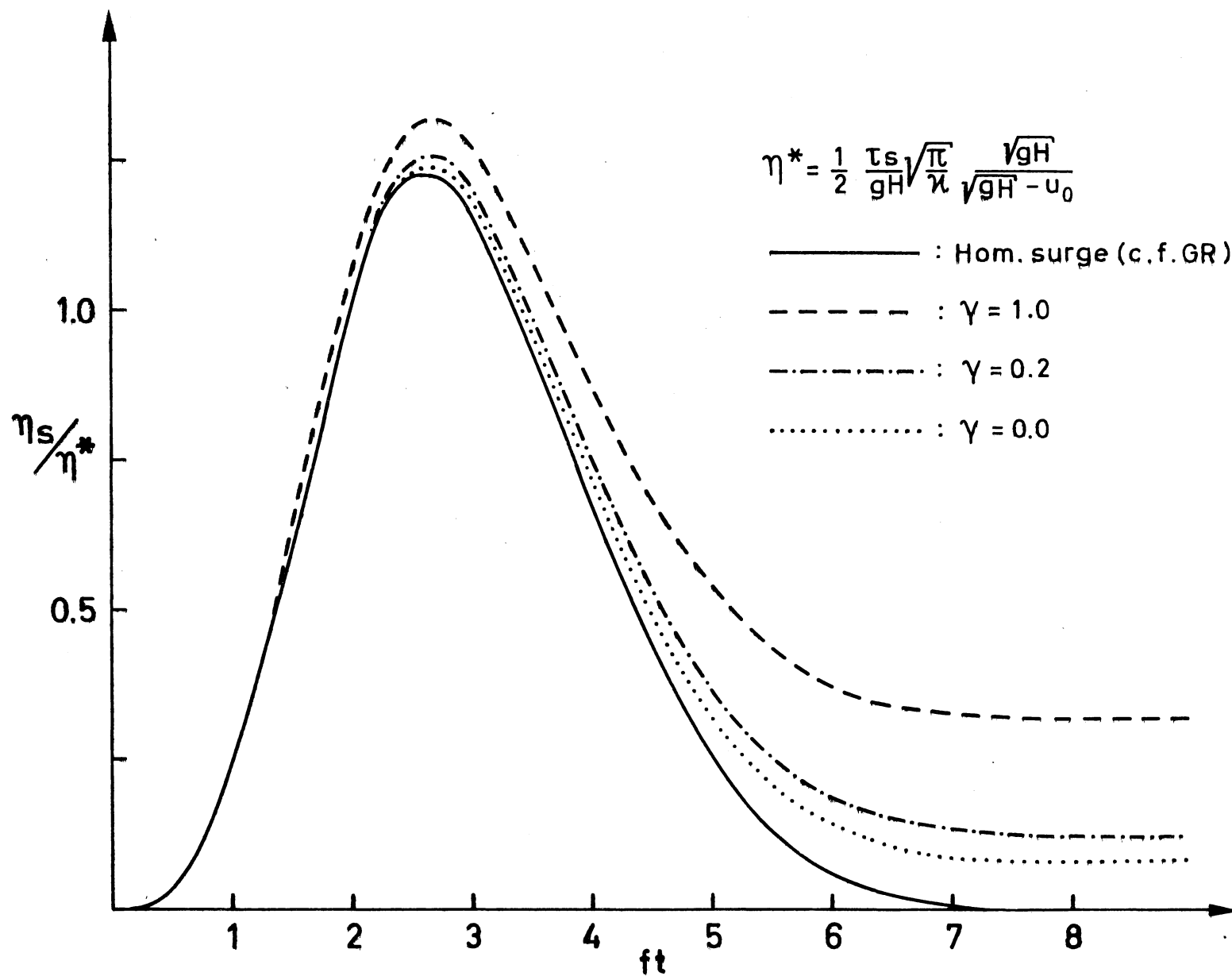


Fig. 4.1. The surge due to a moving storm.

assumption may be used to decide which of the three stress models are the most appropriate. Fig. 4.1 reveals that the best fit to the concomitant homogeneous surge is made with $\gamma = 0$ i.e. the stress model based on an eddy viscosity coefficient inversely proportional to the density gradient.

5. The effect of bottom friction

The form of the modes equations (2.15) and (2.16) suggests that the displacement due to each mode separately corresponds to the surge derivable from a model with a homogeneous body of water in a basin of depth

$$(5.1) \quad H_n = \frac{c_n^2}{g}$$

(c.f. Csanady, Lighthill). Thus for the barotropic mode this depth approximates the real depth ($H_0 \approx H$), whereas, for instance, for the first baroclinic mode this depth is 5 cm only (c.f. table 1). For the higher baroclinics the depths are even smaller. Gammelsrød et.al. (1975) have suggested a procedure whereby the time scales for the influence of the bottom friction in homogeneous models may be ascertained. They suggest that the time scales are proportional to the depth of the basin, viz.

$$\text{frictional time scales} \approx \frac{2\pi H}{fD}$$

where D is some frictional depth (for instance the Ekman depth). For the barotropic mode this entails $ft \sim 10-15$ depending on the ratio H_0/D . For ocean basins of depth less than D the procedure is less conclusive, but the author believes that for depth pertinent

to the baroclinic modes bottom friction is important already after a short elapse of time, $ft < 1$ say. Thus in relatively deep coastal areas scaled on D the effect of bottom friction on the barotropic mode is negligible except, perhaps, for storms of unusual long duration, whereas for the baroclinic modes bottom friction has to be included immediately. This conclusion is supported by the remarkable large internal displacements found by Csanady, which also invalidate the linear theory assumption already after a short elapse of time.

According to Mork the modes equations Eqs. (2.15) and (2.16) including bottom friction read

$$(5.2) \quad \left(\frac{\partial}{\partial t} + R_n\right) \vec{v}_n + f\vec{k} \times \vec{v}_n = -c_n^2 \nabla \eta_n + \vec{\tau}_n$$

$$(5.3) \quad \nabla \cdot \vec{v}_n + \frac{\partial \eta_n}{\partial t} = 0$$

where R_n is given by

$$(5.4) \quad R_n = \lambda \frac{gH}{c_n^2}$$

Assuming friction to be negligible for the barotropic mode it follows from Eq. (5.2) that the term $R_0 \vec{v}_0$ should be small compared to $\partial \vec{v}_0 / \partial t$. Thus the barotropic mode equation is unchanged. However, anticipating friction to be important for the baroclinic modes, one might infer that

$$(5.5) \quad R_n \vec{v}_n \gg \frac{\partial \vec{v}_n}{\partial t} \quad ; \quad n \geq 1$$

which change the equation governing the baroclinic modes to

$$(5.6) \quad R_n \vec{v}_n + f\vec{k} \times \vec{v}_n = -c_n^2 \nabla \eta_n + \vec{\tau}_n$$

$$(5.7) \quad \nabla \cdot \vec{v}_n + \frac{\partial \eta_n}{\partial t} = 0$$

The pertinent time scales for storm surge problems are a couple of hours and a scaling of $\partial/\partial t$ by f is appropriate. Thus Eq. (5.5) may be interpreted as

$$(5.8) \quad R_n/f \gg 1 \quad ; \quad n \geq 1$$

Now, for instance, for the first baroclinic mode

$$(5.9) \quad gH/c_n^2 = 5.3279 \cdot 10^3 \quad ; \quad n = 1.$$

Thus for the inequality (5.8) to hold for all $n \geq 1$

$$(5.10) \quad \lambda/f \gg 1.8769 \cdot 10^{-4}$$

Furthermore, since $R_0 \ll f$ from the barotropic mode λ/f is confined by

$$(5.11) \quad 1.8769 \cdot 10^{-4} \ll \frac{\lambda}{f} \ll 1.$$

Thus an appropriate value of λ/f is 10^{-2} .

From Eqs. (5.6) and (5.7) the following equation for displacement of each baroclinic mode may be derived :

$$(5.12) \quad c_n^2 \nabla^2 \eta_n - R_n \frac{\partial \eta_n}{\partial t} = \nabla \cdot \vec{\tau}_n \quad ; \quad n \geq 1.$$

Application of the boundary condition at the coast, Eq. (2.19), yields the condition

$$(5.13) \quad c_n^2 \left(f \frac{\partial \eta_n}{\partial x} - R_n \frac{\partial \eta_n}{\partial y} \right) = f \vec{i} \cdot \vec{\tau}_n \quad ; \quad y = 0.$$

The baroclinic modes do have variations normal to the coast greatly in excess of the variations along the coast (Charney, Gill and Clarke

and others). Hence in the vicinity of the coast Eqs. (5.12) and (5.13) may be approximated by

$$(5.14) \quad c_n^2 \frac{\partial^2 \eta_n}{\partial y^2} - R_n \frac{\partial \eta_n}{\partial t} = \nabla \cdot \vec{\tau}_n$$

$$(5.15) \quad c_n^2 R_n \frac{\partial \eta_n}{\partial y} = -f \vec{i} \cdot \vec{\tau}_n \quad ; \quad y = 0 ,$$

which is a diffusion equation for the slope normal to the coast. The solution to Eq. (5.14) is well known and the contribution from the baroclinic modes, with bottom friction included, to the displacement at the coast, invoking the stress distribution of section 4, may be written

$$(5.16) \quad \eta_{\text{baroc}} = \frac{1}{2} \frac{\tau_0}{gH} \sqrt{\frac{\pi}{\kappa}} g(x,t) \sum_{n=1}^{\infty} \frac{c_n^2}{gH} s_n(\gamma) f_n(z)$$

where

$$(5.17) \quad g(x,t) = \frac{\sqrt{gH}}{\lambda} \sqrt{\frac{\kappa}{\pi}} \left\{ \frac{2f}{\sqrt{\pi\lambda}} \int_0^{\sqrt{t}} e^{-\kappa(x-u_0 t + u_0 \xi^2)^2} d\xi \right. \\ \left. + \frac{\sqrt{gH}}{u_0} \left[e^{-\kappa(x-u_0 t)^2} - e^{-\kappa x^2} \right] \right\}$$

Since from Eqs. (A.8) and (A.16) the product $c_n^2 s_n(\gamma)$ dies off as n^{-4} as n increases for $n \geq 5$ there follows that the series in Eq. (5.16) may be truncated already after a few terms, 6-7 say.

6. Summary and final remarks.

The storm surge in a stratified coastal sea due to the passage of a single storm has been studied. The main conclusions arrived at are :

- 1) The inclusion of the baroclinic modes has only a minor effect on the storm surge.
- 2) Bottom friction seems to have no appreciable effect on the storm surge in relatively deep seas within the lifetime of the surge due to reasonable storm durations.
- 3) The internal response, however, is strongly influenced both by stratification and by bottom friction. Moreover, the internal response is sensitive to the tangential stress dependency on depth.

For prediction purposes, therefore, the storm surge is obtained with sufficient accuracy from homogeneous models. Furthermore, if the coastal sea is relatively deep, i.e. the depth scaled on the Ekman depth is a large number, bottom friction may be neglected too, except possibly for storms of unusually long duration, i.e. storms which have not subsided after approximately one to two days.

However, normally homogeneous models have the shortcoming that they give no information about the vertical structure of the motion. (The vertical structure may be retained in homogeneous models by a method described in Heaps, 1971). To this end stratification must be included and the question arises how the tangential stress are distributed. In fact, solutions of the upwelling problems hinge upon such a knowledge. The storm surge might be a helpful guide in assessing the stress distribution, because it is nearly unaffected by the stratification and might therefore be taken as known, leaving the stress distribution as the primary unknown.

Appendix

On account of the density model chosen in section 3 the Väisälä-Brunt frequency, Eq. (2.13), is zero in the well mixed layer and in the deep bottom layer, and constant throughout the pycnocline. In order to find the eigenfunction, $\hat{\eta}_n(z)$, and eigenvalues, c_n , one has to solve Eq. (2.9) subject to the conditions (2.10) and (2.11). At the levels $z = -h_1$ and $z = -h_2$ describing the bottom of the well mixed layer and the pycnocline layer, respectively, the Väisälä-Brunt frequency is discontinuous, which necessitates conditions to be imposed on $\hat{\eta}_n$ at these levels. According to Eqs. (2.6) and (2.7) the requirement of continuous velocity and displacements entails continuity of $\hat{\eta}'_n$ and $\hat{\eta}_n$ as well. Invoking the continuity requirements Eqs. (2.9) and (2.11) yield

$$(A.1) \quad \hat{\eta}_n(z) = a_n f_n(z) ,$$

where

$$(A.2) \quad f_n(z) = \begin{cases} \zeta + \left(\frac{F_1}{\alpha_n}\right)^2 & ; -\delta_1 \leq \zeta \leq 0 \\ \frac{\delta_2 - \delta_1}{\alpha_n} \left[\sin(\alpha_n \frac{\zeta + \delta_1}{\delta_2 - \delta_1}) + B_n \cos(\alpha_n \frac{\zeta + \delta_1}{\delta_2 - \delta_1}) \right] & ; -\delta_2 < \zeta < -\delta_1 \\ (\cos \alpha_n + B_n \sin \alpha_n)(1 + \zeta) & ; -1 \leq \zeta \leq -\delta_2 . \end{cases}$$

Here F_1 is the internal Froude number

$$(A.3) \quad F_1 = \frac{NH}{\sqrt{gH}} (\delta_2 - \delta_1)$$

and

$$(A.4) \quad \delta_1 = \frac{h_1}{H} ; \quad 1 = 1, 2 , \quad \zeta = \frac{z}{H} .$$

The α_n 's, which are solutions of the transcendental equation

$$(A.5) \quad \operatorname{tg} \alpha_n = \frac{(\delta_2 - \delta_1)}{\alpha_n} \frac{F_1^2 - (1 - \delta_2 + \delta_1) \alpha_n^2}{(\delta_2 - \delta_1)^2 + F_1^2 (1 - \delta_2) - \delta_1 (1 - \delta_2) \alpha_n^2} = h(\alpha) ,$$

are related to the eigenvalues, c_n , through the relation

$$(A.6) \quad \frac{c_n}{\sqrt{gH}} = \frac{F_1}{\alpha_n} .$$

Finally B_n is defined by the expression

$$(A.7) \quad B_n = \frac{\alpha_n}{\delta_2 - \delta_1} \left[\left(\frac{F_1}{\alpha_n} \right)^2 - \delta_1 \right] .$$

The roots of Eq. (A.5) may be visualized with the help of Fig. A.1. Here the right hand side of Eq. (A.5), $h(\alpha)$, is drawn together with $\operatorname{tg} \alpha$. For increasing values of n it is seen that α_n rapidly approaches $(n-1)\pi$. Hence a very good approximation for c_n for moderate values of n , $n \geq 5$, say, is

$$(A.8) \quad c_n \approx \frac{NH}{(n-1)\pi} (\delta_2 - \delta_1)$$

Furthermore, the zeroth mode is close to zero which entails

$$(A.9) \quad \operatorname{tg} \alpha_0 \approx \alpha_0 .$$

Thus from Eq. (A.5) one obtains

$$(A.10) \quad \left(\frac{F_1}{\alpha_0} \right)^2 \approx 1$$

where use has been made of the fact that $F_1 \ll 1$.

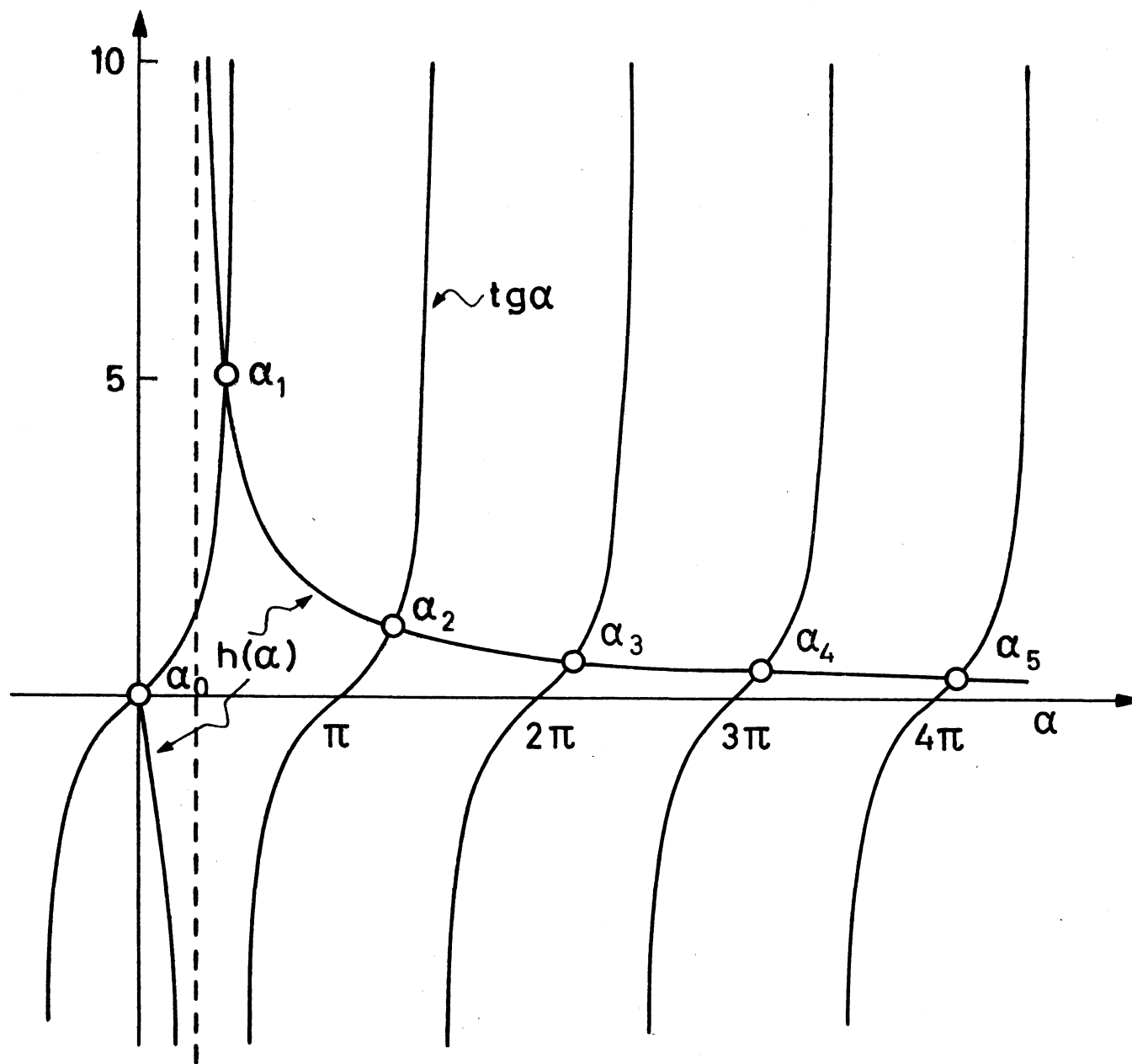


Fig. A.1. Roots, α_n , of the transcendental Eq. (A.5)

Invoking Eq. (A.6) it follows that

$$(A.11) \quad c_0 \approx \sqrt{gH}$$

in accordance with previous investigations. The amplitudes of the eigenfunctions, a_{n1} , are determined from the normalization condition Eq. (2.14) which leads to the expression

$$(A.12) \quad \left(\frac{H}{a_{n1}}\right)^2 = 1 - \frac{1}{2}(\delta_2 - \delta_1)(1 - B_n^2) + \left[(1 - \delta_2)B_n + \frac{\delta_2 - \delta_1}{4\alpha_n}(1 - B_n^2)\right]\sin 2\alpha_n \\ + \left[\frac{\delta_2 - \delta_1}{\alpha_n} B_n - (1 - \delta_2)(1 - B_n^2)\right]\sin^2 \alpha_n \quad .$$

Moreover, the expression (4.11) for $s_n(\gamma)$ may now be written

$$(A.13) \quad s_n(\gamma) = \left(\frac{a_{n1}}{H}\right)^2 \left\{ \gamma + \frac{1-\gamma}{\alpha_n} \left[B_n(1 - \cos \alpha_n) + \sin \alpha_n \right] \right\} \quad .$$

For $n \geq 5$ the following approximations may be made

$$(A.14) \quad B_n \approx - \frac{(n-1)\pi\delta_1}{\delta_2 - \delta_1}$$

$$(A.15) \quad \left(\frac{H}{a_{n1}}\right) \approx 1 + \frac{1}{2} \frac{(n-1)^2 \pi^2 \delta_1^2}{\delta_2 - \delta_1}$$

$$(A.16) \quad s_n(\gamma) \approx \frac{\gamma(\delta_2 - \delta_1) - (1-\gamma)\delta_1[1 + (-1)^n]}{(\delta_2 - \delta_1) + \frac{1}{2}(n-1)^2 \pi^2 \delta_1^2} \quad .$$

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